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1983 J. Phys. A: Math. Gen. 16 1175

(http://iopscience.iop.org/0305-4470/16/6/012)

J. Phys. A: Math. Gen. 16 (1983) 1175–1180. Printed in Great Britain

# Colliding plane gravitational and electromagnetic waves

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Received 4 October 1982

**Abstract.** A new class of type-II Einstein–Maxwell fields is presented. These describe the space–time subsequent to a collision of a gravitational wave with an electromagnetic wave. The gravitational wave is an arbitrary pp wave with generally non-constant polarisation. The incoming electromagnetic wave is also an arbitrary pp wave, and it is shown that this is partially reflected by the gravitational wave.

## 1. Introduction

An interesting feature of Einstein's general theory of relativity is that, being a nonlinear theory, it does not allow a simple superposition of fields. An exception to this occurs for pp waves propagating in the same direction. Bonnor (1969) and Aichelburg (1971) have shown that in this case the field equations are linear when written in a certain privileged class of coordinate systems, so that gravitational and electromagnetic waves can be simply superposed. In the general case, however, a nonlinear interaction occurs.

The effects of this nonlinear interaction can be seen most clearly in situations in which two distinct waves, moving in different directions, propagate into the same region. In such cases it may be assumed that the background field, and the two waves before they collide, are all known. The problem is then to solve the field equations in the interaction region subsequent to the collision, and to examine the solutions.

This procedure has been considered by a number of authors. However, only a few particular exact solutions have yet been obtained. Attention has so far been restricted to the collision of pp waves, and these have generally been considered in a flat background. When considering pp waves, it is of course always possible to choose a frame of reference in which the two waves propagate in directly opposite directions, so that only 'head on' collisions need to be considered.

Szekeres (1972) has obtained a class of exact solutions describing the collision and interaction of pp gravitational waves with certain profiles. However, these solutions are also restricted by the condition that the incoming waves must have aligned constant polarisation. Ray (1980), Nutku and Halil (1977) and Halil (1979) have obtained solutions in which the polarisation of the incoming waves is not aligned, although they still have constant polarisation and particular profiles. A particular solution in which the incoming waves have variable polarisation has been obtained by Panov (1979b). Some general properties of solutions of this type have been described by Sbytov (1976) and Tipler (1980). Bell and Szekeres (1974) have also presented a solution for the collision and interaction of two constant-profile electromagnetic pp waves. Other examples of colliding electromagnetic and gravitational waves have been obtained by Panov (1978, 1979a). Particular solutions describing the collision and scattering of an electromagnetic wave by a gravitational wave have been given by Griffiths (1976b). The problem of colliding neutrino fields has also been considered by Griffiths (1976a), and colliding scalar waves by Wu (1982). Finally, particular solutions describing colliding gravitational waves in an expanding background have been obtained by Centrella and Matzner (1982).

The purpose of the present paper is to present a new class of exact solutions which describe the collision and interaction of a gravitational wave and an electromagnetic wave. In contrast to previous solutions of this type, these solutions are totally general. They describe the collision of an arbitrary pp gravitational wave with generally non-constant polarisation, and an arbitrary pp electromagnetic wave. The metric in the interaction region describes a new class of type-II Einstein–Maxwell fields.

## 2. Field equations and boundary conditions

The situation of a collision and subsequent interaction of two pp waves can best be described in terms of four distinct regions of space–time. These are connected along two null hypersurfaces $u = 0$ and $v = 0$, which in this case represent the wavefronts of the gravitational wave and electromagnetic wave respectively. This situation is represented in figure 1.

It is assumed that throughout all four regions, two null coordinates $u$ and $v$, and two space-like coordinates $x$ and $y$ can be chosen for convenience. The metric in each region is chosen in such a way that the Lichnerowicz conditions are satisfied on each boundary. However, these must be relaxed in favour of the O'Brien–Synge conditions if impulsive gravitational waves are included (Robson 1973). It is well known that if the metrics in regions I, II and III are initially specified then the space–time in region IV is determined uniquely. We therefore proceed to specify the fields in regions I, II and III and give field equations and boundary conditions for the metric in region IV.

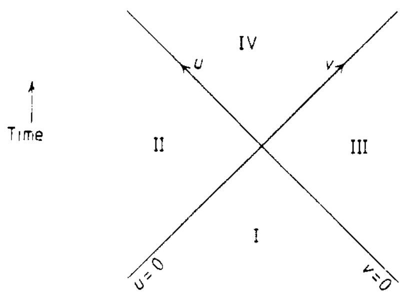


**Figure 1.** Region I is the background which in this case is taken to be flat. Regions II and III contain approaching gravitational and electromagnetic waves respectively. Region IV represents the interaction region subsequent to the collision at $u = 0$, $v = 0$. Null coordinates $u$ and $v$ are chosen for convenience.

### 2.1. *Region I:* $u < 0$, $v < 0$

This background region is taken to be flat, and its metric can be written in the form

$$ds^2 = 2\, du\, dv - dx^2 - dy^2.$$

### 2.2. *Region II: $u \geq 0$, $v < 0$*

This region is taken to contain a general pp gravitational wave with wavefront given by $u = 0$. The metric can be expressed in harmonic coordinates as

$$ds^2 = 2\,du\,dV + [A(X^2 - Y^2) + 2BXY]\,du^2 - dX^2 - dY^2 \tag{1}$$

where $A$ and $B$ are two arbitrary functions of $u$. It is well known that if $A/B$ is a constant, then the wave is said to have constant polarisation, and it is possible to find a coordinate transformation which reduces the term $B$ to zero. However, this condition is not required here.

It has been pointed out by Szekeres (1972) that, for situations of the type considered here, it is more convenient to transform the metric (1) into Rosen form

$$ds^2 = 2\,du\,dv - (\tfrac{1}{2}+f)(e^V \cosh W\,dx^2 + e^{-V}\cosh W\,dy^2 - 2\sinh W\,dx\,dy) \tag{2}$$

where $f$, $V$ and $W$ are functions of $u$, and are related by the single equation

$$W_u^2 + V_u^2 \cosh^2 W + 2f_{uu}(\tfrac{1}{2}+f)^{-1} - f_u^2(\tfrac{1}{2}+f)^{-2} = 0 \tag{3}$$

where for later convenience the derivative of a function is denoted by a subscript.

In order to satisfy appropriate boundary conditions it is also assumed that

$$f = \tfrac{1}{2} \qquad V = W = f_u = V_u = W_u = 0 \quad \text{when} \quad u = 0.$$

It can thus be seen that the space–time in regions I and II describes a general pp gravitational wave, with wavefront on the null hypersurface $u = 0$, propagating into a region that is initially flat.

### 2.3. *Region III: $u < 0$, $v \geq 0$*

This region is taken to contain a general pp electromagnetic wave with wavefront given by $v = 0$. It is assumed that there is no associated free gravitational wave, so that the metric is asymptotically flat. This can be expressed in harmonic coordinates as

$$ds^2 = 2\,dv\,dU + C(X^2 + Y^2)\,dv^2 - dX^2 - dY^2$$

where $C$ is an arbitrary function of $v$. Again it is convenient to transform this into Rosen form

$$ds^2 = 2\,e^{-M}\,du\,dv - (\tfrac{1}{2}+g)(dx^2 + dy^2) \tag{4}$$

where $g$ and $M$ are functions of $v$. The electromagnetic wave is described by the component $\Phi_0$ using a scale-invariant form of the Newman–Penrose notation, where

$$4\Phi_0\bar{\Phi}_0 = -2g_{vv}(\tfrac{1}{2}+g)^{-1} + g_v^2(\tfrac{1}{2}+g)^{-2} - 2g_vM_v(\tfrac{1}{2}+g)^{-1}. \tag{5}$$

It is, of course, possible to use a particular choice of the $v$ coordinate such that $M = 0$. In such a case $\Phi_0$ would be determined up to an arbitrary phase by the function $g(v)$ according to equation (5). For later convenience, however, it is preferable here to retain the additional arbitrariness, and at this stage to regard $g$, $\Phi_0$ and $M$ as functions of $v$ that are only required to satisfy equation (5).

In order to satisfy the appropriate boundary conditions, it is assumed that

$$g = \tfrac{1}{2} \qquad M = M_v = g_v = 0 \quad \text{when} \quad v = 0.$$

*2.4. Region IV: $u \geqslant 0$, $v \geqslant 0$*

This region describes the interaction between the gravitational and electromagnetic waves after their collision. It can be shown (Szekeres 1972) that the metric in this region can be taken in the general form

$$ds^2 = 2\,e^{-M}\,du\,dv - e^{-U}(e^V \cosh W\,dx^2 + e^{-V}\cosh W\,dy^2 - 2\sinh W\,dx\,dy) \tag{6}$$

where $M$, $U$, $V$ and $W$ are all functions of the two null coordinates $u$ and $v$. The field equations can be quoted directly from Bell and Szekeres (1974) or Griffiths (1967b), using the same notation. They are

$$U_{uv} = U_u U_v \tag{7}$$

$$2U_{vv} = U_v^2 + W_v^2 + V_v^2 \cosh^2 W - 2U_v M_v + 4\Phi_0\bar{\Phi}_0 \tag{8}$$

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2 \cosh^2 W - 2U_u M_u + 4\Phi_2\bar{\Phi}_2 \tag{9}$$

$$2W_{uv} = U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W + 2\mathrm{i}(\Phi_0\bar{\Phi}_2 - \Phi_2\bar{\Phi}_0) \tag{10}$$

$$2V_{uv}\cosh W = (U_u V_v + U_v V_u)\cosh W$$
$$- 2(V_v W_u + V_u W_v)\sinh W + 2(\Phi_0\bar{\Phi}_2 + \Phi_2\bar{\Phi}_0) \tag{11}$$

$$2M_{uv} = -U_u U_v + W_u W_v + V_u V_v \cosh^2 W \tag{12}$$

$$\Phi_{2,v} = \tfrac{1}{2}(U_v + \mathrm{i}V_v \sinh W)\Phi_2 - \tfrac{1}{2}(\mathrm{i}W_u + V_u \cosh W)\Phi_0 \tag{13}$$

$$\Phi_{0,u} = \tfrac{1}{2}(U_u - \mathrm{i}V_u \sinh W)\Phi_0 + \tfrac{1}{2}(\mathrm{i}W_v - V_v \cosh W)\Phi_2 \tag{14}$$

where equations (13) and (14) are, in fact, Maxwell equations.

It is, of course, also required that the functions $M$, $U$, $V$ and $W$ should be continuous and smooth across the boundaries $u = 0$ and $v = 0$; also $\Phi_0$ must be continuous across $u = 0$, and $\Phi_2$ zero on $v = 0$. These functions are all determined on these hypersurfaces in terms of the functions associated with the incoming waves. The problem now remains to solve these equations, subject to the boundary conditions, and to interpret them.

## 3. A general solution in the interaction region

It can immediately be seen from (7) and the appropriate boundary conditions that the function $U$ must be given by

$$e^{-U} = f(u) + g(v) \tag{15}$$

where $f$ and $g$ are the same functions as appear in regions II and III respectively.

It is now appropriate to consider the possibility that $V$ and $W$ also retain the same functional form as in region II. This suggestion does, in fact, always lead to exact solutions, and since the space–time in region IV is uniquely determined, it is appropriate to apply the condition that $V$ and $W$ are independent of $v$.

$$V = V(u) \qquad W = W(u). \tag{16}$$

With this condition, equations (10) and (11) imply that

$$\Phi_0\bar{\Phi}_2 = \tfrac{1}{4}g_v(V_u \cosh W - \mathrm{i}W_u)(f+g)^{-1}.$$

Equations (13), (14), (10) and (11) then imply that

$$\Phi_0 = \tfrac{1}{2} g_v (f+g)^{-1/2} (\tfrac{1}{2} - g)^{-1/2} \, e^{i\theta} \tag{17}$$

$$\Phi_2 = \tfrac{1}{2} (V_u \cosh W + i W_u)(f+g)^{-1/2} (\tfrac{1}{2} - g)^{1/2} \, e^{i\theta} \tag{18}$$

where $\theta$ is a function of $u$ only, that must satisfy the equation

$$\theta_u = -\tfrac{1}{2} V_u \sinh W. \tag{19}$$

This then determines the electromagnetic field in terms of the given functions up to an arbitrary constant phase.

Finally, equations (8) and (9) can be integrated to obtain

$$e^{-M} = g_v (f+g)^{-1/2} (\tfrac{1}{2} + f)^{1/2} (\tfrac{1}{2} - g)^{-1/2}. \tag{20}$$

With this the remaining equation (12) is automatically satisfied.

The metric and the electromagnetic field in region IV are now determined, and it can be shown that the scale-invariant components of the Weyl tensor only have the non-zero components

$$\Psi_4 = -\tfrac{1}{2}(i W_{uu} + V_{uu} \cosh W) - V_u W_u \sinh W + \tfrac{1}{2} i V_u^2 \sinh W \cosh W$$
$$- \tfrac{1}{4} f_u [3(f+g)^{-1} - (\tfrac{1}{2} + f)^{-1}](i W_u + V_u \cosh W)$$
$$\Psi_2 = -\tfrac{1}{4} f_u g_v (f+g)^{-2}$$

which indicates that the field is of algebraic type II.

## 4. Interpretation of the solution

In the solution given above, the incoming gravitational wave is a general type-N pp wave, the curvature tensor having only a $\Psi_4$ component. It is described by three functions, $f(u)$, $V(u)$ and $W(u)$, which are connected only through the single equation (3).

The opposing electromagnetic wave, prior to the collision, is also a general pp wave with curvature tensor having only a $\Phi_{00}$ component. The metric is given by (4), which contains the arbitrary function $g(v)$. The electromagnetic field is defined by the component

$$\Phi_0 = \tfrac{1}{2} g_v (\tfrac{1}{2} + g)^{-1/2} (\tfrac{1}{2} - g)^{-1/2} \, e^{i\alpha}$$

where $\alpha$ is a constant. Also the function $M(v)$, which appears in the metric (4) for convenience, is given by

$$e^{-M} = g_v (\tfrac{1}{2} + g)^{-1/2} (\tfrac{1}{2} - g)^{-1/2}.$$

With these choices, equation (5) is satisfied automatically.

The interaction region is determined by the metric form (6) which with (15), (16) and (20) is specified by the functions $f(u)$, $V(u)$ and $W(u)$ that determine the incoming gravitational wave, and $g(v)$ which determines the incoming electromagnetic wave. Thus equation (3) must also be satisfied in this region. It can be seen that $f$ and $g$ are both decreasing functions, and that the metric becomes singular on the space-like hypersurface $f+g=0$.

The electromagnetic wave component $\Phi_0$ continues through the interaction region. However, the component $\Phi_2$ also appears, which indicates that the incoming wave has been partially reflected. This effect is expected (Penrose 1972), since it is known by the Mariot–Robinson theorem that a null electromagnetic wave necessarily follows a shear-free null geodesic congruence. In this case the congruence associated with the component $\Phi_0$ has non-zero shear, so scattering is expected.

The gravitational wave component $\Psi_4$ also continues through the interaction region, in which the additional component $\Psi_2$ also appears. It can now be seen from the curvature invariants

$$I_1 - iI_2 = 16\,e^{2M}(3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4)$$

that the singularity on the hypersurface $f+g=0$ is an essential curvature singularity, of the same type as those encountered with colliding gravitational waves (Tipler 1980).

Finally it must be pointed out that the solution given above for the interaction region defines a new class of Einstein–Maxwell fields. They are of algebraic type II. However, in this case the components of the Ricci tensor $\Phi_{00}$ and $\Phi_{02}$ are both non-zero, which immediately indicates that they do not belong to the classes of algebraically special Einstein–Maxwell fields that have previously been considered. The class of solutions obtained here has been presented in terms of an immediate physical interpretation. In addition, they include the solutions given previously (Griffiths 1976b) in which the electromagnetic wave has constant profile and the gravitational wave has constant polarisation.

## Acknowledgment

I am grateful to Mr N H E Prince for many discussions of this problem, and for checking the calculations.

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